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The last remake of the mixmaster universe model

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Abstract. We review the existing evidence on the (non)integrability of the mixmaster universe model. We show how a local Painlevé analysis can be used to study the possible existence of essential singularities. In agreement with recent studies, we find that the mixmaster model possesses critical essential singularities, the multivalued character of which is incompatible with integrability. Our analysis is complemented by a numerical study of the spectrum of the stretching numbers of the system. We show that the zero-energy case is characterized by a vanishing maximal Lyapunov characteristic number.

1. Introduction

The mixmaster universe model (MUM) [1, 2] has attracted the interest of both cosmologists and integrability specialists because of its somewhat ambiguous dynamical behaviour. In particular, and contrary to what is initially expected [3, 4], numerical simulations have failed to exhibit ergodicity [5–8]. The absence of chaotic character led naturally to the consideration of the possible integrability of this model [9, 10]. As we shall show in what follows this was too optimistic a view and the recent analytical results suggest non-integrability. The mixmaster universe model, also known as the Bianchi IX model, is obtained through the solution of Einstein's equations and corresponds to a Hamiltonian system in three dimensions with zero energy [11]. The equations of motion read

$$2\ddot{\alpha} = (e^{2\beta} - e^{2\gamma})^2 - e^{4\alpha} \qquad 2\ddot{\beta} = (e^{2\gamma} - e^{2\alpha})^2 - e^{4\beta} \qquad 2\ddot{\gamma} = (e^{2\alpha} - e^{2\beta})^2 - e^{4\gamma} \tag{1}$$

where the dots mean derivatives with respect to time t. The zero-energy condition in the α, β, γ variables is

$$E = -4(\dot{\alpha}\dot{\beta} + \dot{\beta}\dot{\gamma} + \dot{\gamma}\dot{\alpha}) + e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2e^{2(\alpha+\beta)} - 2e^{2(\beta+\gamma)} - 2e^{2(\gamma+\alpha)} = 0. \tag{2}$$

In [12] we introduced a convenient variable transformation that allows us to write the MUM equations of motion in a purely polynomial form:

$$\begin{aligned} &X = e^{2\alpha} \qquad Y = e^{2\beta} \qquad Z = e^{2\gamma} \\ &p_x = -(\dot{\beta} + \dot{\gamma}) \qquad p_y = -(\dot{\gamma} + \dot{\alpha}) \qquad p_z = -(\dot{\alpha} + \dot{\beta}). \end{aligned} \tag{3}$$

We must point out that this transformation is not canonical, i.e. the p's are not the conjugate momenta of the X's, however, it is still straightforward to transcribe (1) into the new variables. We find

$$\dot{X} = X(p_x - p_y - p_z) \tag{4a}$$
$$\dot{Y} = Y(p_y - p_z - p_x) \tag{4b}$$
$$\dot{Z} = Z(p_z - p_x - p_y) \tag{4c}$$
$$\dot{p}_x = X(Y + Z - X) \tag{4d}$$
$$\dot{p}_y = Y(Z + X - Y) \tag{4e}$$
$$\dot{p}_z = Z(X + Y - Z) \tag{4f}$$

while the zero-energy condition becomes

$$E = p_x^2 + p_y^2 + p_z^2 - 2p_x p_y - 2p_y p_z - 2p_z p_x + X^2 + Y^2 + Z^2 - 2XY - 2YZ - 2ZX = 0. \tag{5}$$

The aim of the present paper is to review the existing evidence on the integrable or not character of the MUM and present more evidence in favour of non-integrability based on *local* singularity analysis.

2. Brief analysis of the existing results

What do the numerical experiments tell us?

(a) The standard study of the chaotic character of a Hamiltonian system is based on the computation of the maximal Lyapunov characteristic number (LCN). In the case of the MUM the initial calculations have found a positive maximal LCN [13, 14]. However, such calculations are particularly delicate, and when new, more accurate, calculations were performed they led to results compatible with a maximal LCN equal to zero [5–8, 15].

(b) Another numerical experiment is the one of Christiansen *et al* [16]. They studied the stability of the periodic orbits of the model. Having obtained evidence of instabilities they concluded on the non-integrability of the system. However, a word of caution is needed here. Unstable orbits do exist for integrable systems as well, and thus these results cannot constitute a foolproof argument.

(c) A third numerical study of a totally different nature was performed recently by Bountis and Drossos [17]. They studied the position of singularities in the complex-time plane and, in particular, the patterns in which the singularities are organized [18]. The interpretation of such results is not always unambiguous and is based largely on experience. What the authors of [17] find is that all the singularities obtained are poles. However, the singularities accumulate densely and this is considered as an indication of the existence of a natural boundary. Natural boundaries are, of course, not incompatible with integrability but the intricacy of the patterns observed has led Bountis and Drossos to conjecture that the MUM is not integrable.

Thus, the numerical evidence to date is inconclusive, although the more recent results seem to indicate a non-integrable character. In section 4 we shall present new results based on the study of the Lyapunov spectra of orbits of the MUM.

Let us now turn to the analytic results.

(a) A first result is the one due to Sniatycki and Cushman [19]. The argument is indeed elementary. It is based on the observation that, for $E = 0$, the derivative of the quantity

$$\Omega = \frac{1}{XYZ} \tag{6}$$

is a strictly increasing function of time. We find, using equation (4), that

$$\dot{\Omega} = \Omega(p_x + p_y + p_z) \tag{7a}$$

and

$$\ddot{\Omega} = \Omega(2(p_x^2 + p_y^2 + p_z^2) - E) \tag{7b}$$

where E is given by (5). This means that for $E = 0$ and also for $E \leqslant 0$, $\ddot{\Omega}$ has a constant sign (the same as Ω). Thus, if we start with a positive value, we find that Ω decreases from infinity at $\tau = -\infty$ to a minimum and then increases to infinity at $\tau = +\infty$. Hence XYZ goes to zero and recurrences of the orbits are impossible. Thus Sniatycki and Cushman [19] conclude that the system is not chaotic. However, the original variables α, β and γ tend to minus infinity and the system is not compact. In such systems the usual notion of chaos is not applicable. But it is possible that nearby orbits deviate considerably, although the escaping particles move with asymptotically constant velocity along straight lines. In such cases we speak of irregular or chaotic scattering [20]; in section 4 we will present some numerical evidence that this is the case for the MUM.

The remaining results are based essentially on the singularity analysis approach, used under various (hopefully complementary) angles.

(b) In [9] we have presented the Painlevé analysis of the MUM based on the approach of Ablowitz, Ramani and Segur (ARS) [21]. Two singular expansions were identified ($\tau = t - t_0$) [12].

(i) X, p_x alone diverge while Y, Z, p_y, p_z are finite (or any other circular permutation):

$$X = \pm\frac{\mathrm{i}}{\tau} \qquad p_x = -\frac{1}{\tau}$$
$$Y = y_{01}\tau \qquad Z = z_{01}\tau \qquad p_y = p_2 \qquad p_z = p_3$$

with resonances $r = -1, 0, 0, 0, 0, 2$, respectively. This is the 'generic' singular expansion.

(ii) All X, Y, Z, p_x, p_y, p_z diverge as simple poles:

$$X, Y, Z = \pm\frac{\mathrm{i}}{\tau} \qquad p_x, p_y, p_z = \frac{1}{\tau}.$$

The resonances in this case are $r = -1, -1, -1, 2, 2, 2$, reaspectively. One may argue that a triple (-1) resonance indicates the existence of logarithms. However, and we wish to stress this point, this is not the case. As an example in favour of our argument we can present the Halphen system recently analysed by Maciejewski and Strelcyn [22]. This three-degrees of freedom system has a singular behaviour with triple (-1) resonance. Still it has no logarithms and is, in fact, equivalent to the well-known Chazy system [23]. (Anticipating the presentation of case (c) below, we claim that the logarithms enter *only* through the 'interaction' of the resonances -1 and 2.) Thus, following the analysis of [9] the MUM seems to satisfy the ARS criterion.

(c) The presence of these negative resonances is an intriguing fact. This motivated a re-examination of the system in the light of the recommendations of Kruskal. Using a perturbative Painlevé approach (based on an algorithm by Conte *et al* [24]), both the present authors [12] and Latifi *et al* [25] found that incompatibilities appear at the negative resonances. The difficulty here lies in the interpretation of the singular expansions obtained. They are not asymptotic either for times close to the singularity or for times far from it. However, the presence of logarithms (due to the incompatibilities) may be considered as an indication of non-integrability. In fact, Conte *et al* [24] used precisely this approach in order to single out the integrable members of the Chazy family.

(d) Another important piece of evidence is the work of Latifi *et al* [25]. They have performed a perturbative analysis around the Taub solution of the MUM. Starting from this analytically known solution the authors of [25] obtained, through perturbation, essential singularities of sufficient complexity to allow them to proclaim the non-integrability of the MUM. In what follows we shall return to this point and show how the results of Latifi *et al* may be recovered using a purely local singularity analysis.

(e) A last result that must be mentioned here is the one we obtained in [10] using Ziglin's approach [26]. We started by transforming the Hamiltonian of the MUM (through a canonical variable transformation) to

$$H = \tfrac{1}{12}(P_X^2 + P_Y^2 + P_Z^2) + \mathrm{e}^{2\mathrm{i}X}\left[\mathrm{e}^{4Y} + \mathrm{e}^{-2(Y-\sqrt{3}Z)} + \mathrm{e}^{-2(Y+\sqrt{3}Z)} - 2(\mathrm{e}^{-2Y} + \mathrm{e}^{(Y-\sqrt{3}Z)} + \mathrm{e}^{(Y+\sqrt{3}Z)})\right] \tag{8}$$

which is a Hamiltonian of Toda type. Note that the X, P_X here are not the same as in (3) and (4). Next, a particular solution of the equation of motion with $Y = Z = P_Y = P_Z = 0$ was obtained. This led to the equation

$$\ddot{X} = \mathrm{i}\mathrm{e}^{2\mathrm{i}X} \tag{9}$$

that can be integrated in a straightforward way. The crucial step in Ziglin's approach is the one where we consider variations around this particular solution: $X \to X + \xi$, $Y \to 0 + \eta$, $Z \to 0 + \zeta$. The first defines the tangential variational equation (TVE)

$$\ddot{\xi} = -2\mathrm{e}^{2\mathrm{i}X}\xi \tag{10}$$

where X is the solution of (9). The remaining two are the normal variational equations (NVE) and the study of their monodromy properties allows one to conclude on their (non)integrability. Here the normal variational system decouples:

$$\ddot{\eta} = -2\mathrm{e}^{2\mathrm{i}X}\eta \tag{11a}$$

$$\ddot{\zeta} = -2\mathrm{e}^{2\mathrm{i}X}\zeta \tag{11b}$$

and, moreover, the NVE are *identical* to the tangential equation. Since the tangential equation is integrable, the same applies to the normal equations; this is no proof of integrability of course. It just means that *around this particular solution* the system does not exhibit multi-valuedness incompatible with integrability.

To summarize, the analytical results obtained to date seem to converge towards the non-integrability of the MUM. (Let us also mention here that all attempts to obtain constants of motion, besides the Hamiltonian, have failed.) As a further indication we shall present a local singularity analysis that will lead to essential singularities of character incompatible with integrability.

3. Essential singularities from a local Painlevé analysis

In our previous work [9] we examined two singular behaviours of (4) presented as (i) and (ii) in (a) above. The question that can now be asked is whether these two singular behaviours are the only ones. In [9] we argued that no singular behaviour where two of the X's are divergent can exist. However, it turns out that this argument is oversimplified: such a situation can exist (although, admittedly, it is more complicated than initially thought). Let us, thus, assume that two of the X's, say Y and Z, are more singular than X. From equations (4*a*–c) it results that p_x, p_y, p_z diverge like $\mathcal{O}(1/\tau)$. Then (4*d*) shows that Y, Z

must diverge like $1/\tau^2$ while X is regular and starts as a constant. However. this behaviour would be incompatible with (4*e*, *f*) *unless* a cancellation occurs in $Y - Z$. In fact, Y and Z must be equal not only at the level of the dominant term, but also at the subdominant terms of order $1/\tau$. It is then easy to compute the leading singularity:

$$\begin{aligned} &X = A \qquad Y = \frac{B}{\tau^2} \qquad Z = \frac{B}{\tau^2} \\ &p_x = \frac{2}{\tau} \qquad p_y = \frac{1}{\tau} \qquad p_z = \frac{1}{\tau} \end{aligned} \tag{12}$$

with $AB = -1$. The cancellation of the difference $Y - Z$ (and also of $p_y - p_z$) suggests a change of variables where this difference appears explicitly:

$$Y - Z = \delta \qquad Y + Z = \sigma \qquad p_y - p_z = q \qquad p_y + p_z = p. \tag{13}$$

Equations (4) can now be writen as

$$\dot{X} = X(p_x - p) \tag{14a}$$
$$\dot{\sigma} = -p_x\sigma + \delta q \tag{14b}$$
$$\dot{\delta} = -p_x\delta + \sigma q \tag{14c}$$
$$\dot{p}_x = X(\sigma - X) \tag{14d}$$
$$\dot{p} = X\sigma - \delta^2 \tag{14e}$$
$$\dot{q} = (X - \sigma)\delta. \tag{14f}$$

The leading behaviour is $p_x \sim 2/\tau$, $X \sim A$, $p \sim 2/\tau$, $\sigma \sim 2B/\tau^2$. As we have seen previously, δ and q cannot diverge either like $1/\tau^2$ or like $1/\tau$. In fact, the cancellation argument can be taken further by examining more closely equations (14*c*) and (14*f*). Let us first assume that the dominant term in δ is of order τ^n. Then, from (14*c*), we find that the dominant behaviour of q is $\mathcal{O}(\tau^{n+1})$ and using (14*f*) we find that the dominant behaviour in δ is $\mathcal{O}(\tau^{n+2})$. Thus, δ and q vanish at all orders! Going back to the X, p variables we have $Y = Z$, $p_y = p_z$ at all orders.

In order to compute the beyond-all-orders behaviour of δ and q we start by obtaining the singular expansion for the reduced, $\delta = q = 0$, system (14). We have

$$\dot{X} = X(p_x - p) \tag{15a}$$
$$\dot{\sigma} = -p_x\sigma \tag{15b}$$
$$\dot{p}_x = X(\sigma - X) \tag{15c}$$
$$\dot{p} = X\sigma. \tag{15d}$$

The resonances of (15) are -1, 0, 1, 2 corresponding to the expansions

$$\begin{aligned} X &= A + CA^2\tau + \tfrac{1}{2}A^3(C^2 - 1)\tau^2 \\ p_x &= \frac{2}{\tau} + A(C + D\tau) \\ p &= \frac{2}{\tau} + A(D + A)\tau \\ \sigma &= -\frac{2}{A\tau^2} + \frac{2C}{\tau} - AC^2 + D \end{aligned} \tag{16}$$

where A, C and D are free parameters. Next, we remark that (14*c*) and (14*f*) are linear equations in terms of δ and q. Combining the two we can obtain a single second-order equation for δ. From (14*f*) we can drop the subdominant term $X\delta$ and obtain

$$\dot{q} = -\delta\sigma. \tag{17}$$

Eliminating p_x from (14*c*) and (15*b*) we obtain

$$q = \frac{\dot{\delta}}{\sigma} - \frac{\dot{\sigma}\delta}{\sigma^2}. \tag{18}$$

Differentiating this last relation and using (17) we find

$$\ddot{\delta} - 2\dot{\delta}\frac{\dot{\sigma}}{\sigma} + \left(\sigma^2 + 2\left(\frac{\dot{\sigma}}{\sigma}\right)^2 - \frac{\ddot{\sigma}}{\sigma}\right)\delta = 0. \tag{19}$$

Retaining the dominant terms in the expansion of σ we have:

$$\ddot{\delta} + \frac{4}{\tau}\dot{\delta} + \delta\left(\frac{4}{A^2\tau^4} - \frac{8C}{A\tau^3}\right) = 0. \tag{20}$$

Next, we look for the solutions of (20) in the form $\tau^\rho e^{\kappa/\tau}$. We find $\rho = -1 \pm 2C\mathrm{i}$ and $\kappa = \pm 2\mathrm{i}/A$ (where the two $\pm$ signs are *not* independent). The general solution of (20) is

$$\delta = c_1\tau^{2Ci-1}e^{2i/A\tau} + c_2\tau^{-2Ci-1}e^{-2i/A\tau}. \tag{21}$$

Thus, the difference of Y, Z is indeed a quantity beyond all orders of perturbation and contains essential singularities. Moreover, since C is a *free* constant, irrational in general, solution (21) has a transcendental branching point. What is worse, since the general solution contains both terms in (21), there is no possibility of bypassing the transcendental essential singularity by some choice of appropriate Stokes sectors [27]. The conclusion is that the singularity considered leads to critical branching and thus violates the Painlevé property: it is expected to be non-integrable.

Some remarks are in order at this point.

(i) It is clear that the singular behaviour we are studying is the one around the same solution on which Latifi *et al* [25] performed their perturbative calculation. Our results are in perfect agreement with theirs. However, in our case one does not have to know an exact solution in order to perform the calculation: the singular expansion suffices.

(ii) The local singularity analysis can be extended so as to deal with essential singularities. The clue for the existence of essential singularities here was the singular expansion with two missing resonances.

(iii) An alternate way to deal with essential singularities of exponential character was presented by Kruskal in his seminal paper [28]. We do not know of any systematic application of these ideas of Kruskal: if they could be implemented algorithmically they could solve the major difficulty of the detection of essential singularities.

While the MUM is presumably non-integrable there exist interesting reductions that can be integrated. The best known is the Taub metric case [29] corresponding to $Y = Z$, $p_y = p_z$ in the notation of equation (4). The solution in this case can be expressed in terms of elliptic functions. Another reduction of the Bianchi IX model is the one based on the assumption of self-duality [30]. The mixmaster equations of motion reduce, in this case, to the Halphen system (which is also integrable in terms of elliptic functions [31]). Finally, we should like to conclude with a reduction that obeys the constraint of zero energy. Taking $X = \mathrm{i}p_x$, $Y = \mathrm{i}p_y$ and $Z = \mathrm{i}p_z$, we find $E = 0$ and the equations of motion reduce to

$$\dot{X} = X(X - Y - Z) \qquad \dot{Y} = Y(Y - Z - X) \qquad \dot{Z} = Z(Z - X - Y). \tag{22}$$

Introducing the new variables $\Sigma = X + Y + Z$, $P = XY + YZ + ZX$ and $\Pi = XYZ$ we find

$$\dot{\Sigma} = \Sigma^2 - 4P \qquad \dot{P} = -6\Pi \qquad \dot{\Pi} = -\Sigma\Pi. \tag{23}$$

Eliminating Σ and Π, for P we find the equation

$$\dddot{P} = 4P\dot{P} \tag{24}$$

that can be readily integrated once to $\ddot{P} = 2P^2 + \kappa_1$ and then to $\dot{P}^2 = \frac{4}{3}P^3 + 2\kappa_1 P + \kappa_2$, i.e. P is expressed in terms of a Weierstrass elliptic function.

4. The Lyapunov spectra of orbits in the MUM

The LCNs of a system give only an estimate of the asymptotic deviation of nearby orbits:

$$\mathrm{LCN} = \lim_{t\to+\infty} \frac{\ln(\xi(t)/\xi(0))}{t}. \tag{25}$$

Much more detailed information can be found if we take 'local' or 'finite-time' Lyapunov numbers [32, 33]:

$$a = \frac{\ln(\xi(\Delta t)/\xi(0))}{\Delta t}. \tag{26}$$

If we take Δt small, we define a 'short-time Lyapunov number' or 'stretching number'. In the case of maps, we take $\Delta t = 1$, i.e. one iteration of the map [34, 35], while in Hamiltonian systems we can take Δt equal to the integration step [36]. Then we calculate the distribution of the values of a. If after N steps Δt along an orbit, the number of values of a in a given interval $[a, a + \mathrm{d}a]$ is $\mathrm{d}N$, the function

$$S(a) = \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}a} \tag{27}$$

gives the 'spectrum of stretching numbers' or the 'Lyapunov spectrum' of the orbit.

In the case of compact systems the spectrum (for large N) is invariant (a) with respect to the initial conditions along the orbit, (b) with respect to the direction of the initial deviation $\xi(0)$ and (c) with respect to the initial conditions in a connected chaotic domain [35]. However, in a non-compact case the spectrum evolves with the time $t = N\Delta t$.

In the case of the mixmaster model we calculated several spectra with different initial conditions, different time steps Δt, different total times t and different energies E (zero, negative or positive). We used a very accurate integration scheme that gives the successive points in powers of Δt, truncated when the accuracy of every variable is better than a fixed value of 10^{-10} or 10^{-15}. Finally, we calculated the variational equations giving the variations of the variables X, Y, Z, p_x, p_y, p_z (that we call $\mathrm{d}X, \mathrm{d}Y, \mathrm{d}Z, \mathrm{d}p_x, \mathrm{d}p_y, \mathrm{d}p_z$) at the same time as the orbit itself.

A few examples of spectra for $E = 0$ are shown in figure 1. The time interval in figure 1 is $\Delta t = 0.01$. We see that the spectrum is restricted in a small interval around 0, between $a = -0.005$ and 0.10, while outside these limits $S(a)$ is almost exactly zero. For relatively small $t = N\Delta t$ the spectrum has two maxima, one for negative a and one for positive a. However, as the total time t increases, the spectrum becomes more and more peaked around $a = 0$ and it tends to a delta function at $a = 0$. The average value of a is the LCN and this tends to zero as N increases. For total times $t = 10^2$, 10^3 and 10^4 the LCNs are $\mathrm{LCN} = 7 \times 10^{-2}$, 2×10^{-2} and -3×10^{-5}, respectively. Similar results are found if $E < 0$. In all cases the spectrum evolves towards a delta function at $a = 0$.

These calculations are consistent with the result mentioned above, namely that for $E \leqslant 0$ the orbits are non-recurrent. However, the vanishing of the LCN is connected to the fact that the orbits go to $X = Y = Z = 0$, i.e. to minus infinity in the original variables α, β, γ.

In this system nearby orbits deviate with positive local (finite-time) LCNs, and only in the limit $t \to \infty$ does the LCN tend to zero. This behaviour is seen vividly in figures 2(*a*), (*b*)

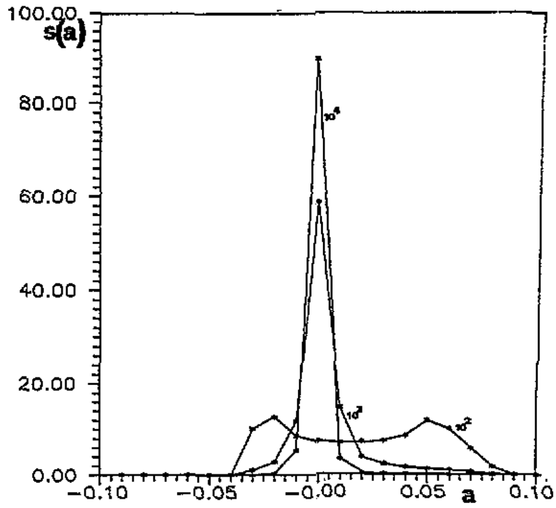

Figure 1. Spectra of the stretching numbers for total times $t = 10^2$, 10^3 and 10^4 time units in a case with $E = 0$. Initial conditions are $(X, Y, Z, p_x, p_y, p_z = 2, 1, 1, 0, 0, 2)$ and $(\mathrm{d}X, \mathrm{d}Y, \mathrm{d}Z, \mathrm{d}p_x, \mathrm{d}p_y, \mathrm{d}p_z = 1, 0, 0, 0, 0, 0)$.

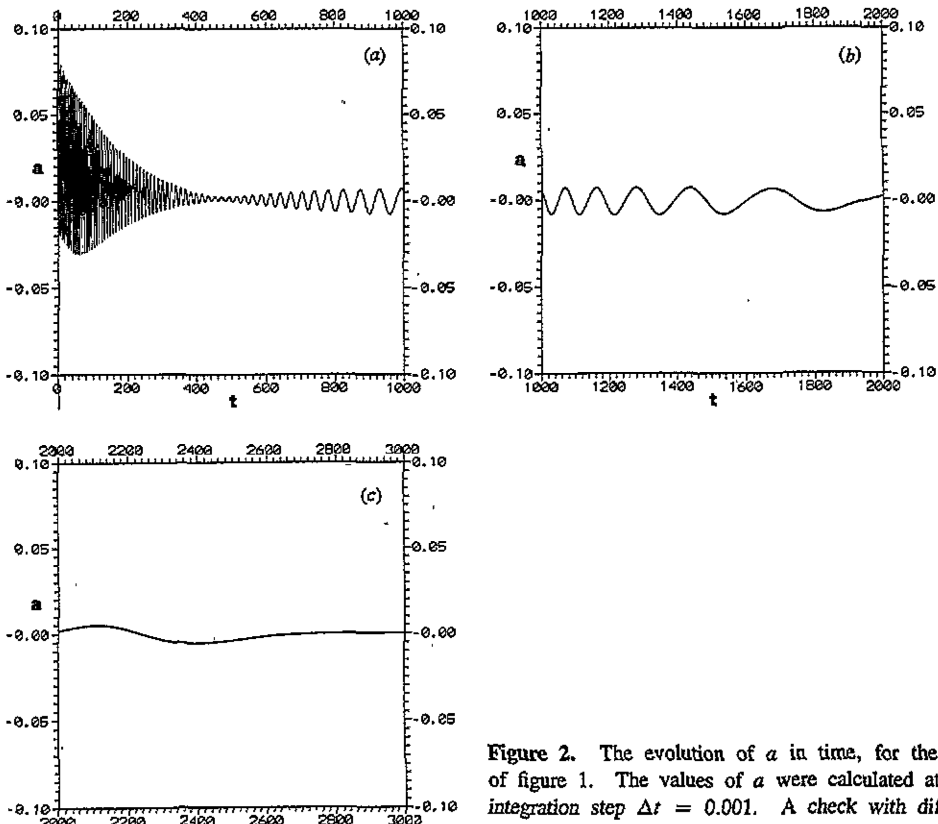

Figure 2. The evolution of a in time, for the orbit of figure 1. The values of a were calculated at each integration step $\Delta t = 0.001$. A check with different integration steps has shown that this figure is exact.

and (c), where we give the stretching number a as a function of time. The values of a have fast oscillations initially, but the average value is clearly positive. Later on, the oscillations become smaller and their period larger. After a long enough time, the oscillations vanish and the value of a tends to zero. This behaviour explains the shrinking of the spectrum as the total time increases (figure 1), and the fact that the finite-time LCN tends to zero when $t \to \infty$.

The case $E > 0$ is more complicated because the value of $2(p_x^2 + p_y^2 + p_z^2) - E$ (equation (7*b*)) changes sign several times and the shrinking of the spectrum is not so fast as in the case $E = 0$. However, this case is only of theoretical interest (it does not represent the MUM) and will not be discussed further here.

We conclude that the existing evidence shows that the mixmaster universe model is not integrable. It cannot be called chaotic in the usual sense, but it has the main properties of a chaotic scattering system.

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